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DIMENSIONAL REDUCTION OF $U(1) \times SU(2)$ CHERN–SIMONS BOSONIZATION: APPLICATION TO THE $t-J$ MODEL

P.A. Marchetti *

Dipartimento di Fisica "G. Galilei"

Istituto Nazionale di Fisica Nucleare, I–35131 Padova, Italy

Zhao–Bin Su

Institute of Theoretical Physics,

Chinese Academy of Sciences, Beijing 100080, China

Lu Yu

International Centre for Theoretical Physics, I-34100 Trieste, Italy

and Institute of Theoretical Physics,

Chinese Academy of Sciences, Beijing 100080, China

ABSTRACT

We perform a dimensional reduction of the $U(1)\times SU(2)$ Chern–Simons bosonization and apply it to the $t-J$ model, relevant for high T_c superconductors. This procedure yields a decomposition of the electron field into a product of two "semionic" fields, i.e. fields obeying abelian braid statistics with statistics parameter $\theta = \frac{1}{4}$, one carrying the charge and the other the spin degrees of freedom. A mean field theory is then shown to reproduce correctly the large distance behaviour of the correlation functions of the 1D $t-J$ model at $t >> J$. This result shows that to capture the essential physical properties of the model one needs a specific "semionic" form of spin–charge separation.

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1. Introduction

It is widely believed that the two–dimensional (2D) $t-J$ model captures many essential physical properties of the $Cu-O$ planes characterizing a large class of high $-T_c$ superconductors.

The hamiltonian of the model is given by

$$H = P_G\Big[\sum_{<ij>} -t(c^\dagger_{i\alpha}c_{j\alpha} + h.c.) + Jc^\dagger_{i\alpha}\frac{\vec{\sigma}_{\alpha\beta}}{2}c_{i\beta}\cdot c^\dagger_{j\gamma}\frac{\vec{\sigma}_{\gamma\delta}}{2}c_{j\delta}\Big]P_G, \tag{1.1}$$

where $c_{i\alpha}$ is the annihilation operator of a spin $\frac{1}{2}$ fermion (in this paper called electron) at site i of a square lattice, corresponding in the physical system to a hole on the Cu site, and P_G is the Gutzwiller projection eliminating double occupation, modelling the strong on–site Coulomb repulsion (see e.g. [1]).

In spite of the enormous efforts made so far, we still do not have a good mean–field theory for the 2D $t-J$ model, i.e. such a successful saddle–point that we can apply the standard quantum many–body techniques to calculate the fluctuations around it for describing the essential physics. On the other hand, we have a much better understanding of the 1D $t-J$ model. In fact, the strictly related large U Hubbard model has been solved by Bethe–Ansatz [2] and for $U \sim +\infty$ the ground state wave function can be written [3] as a product of a Slater determinant for spinless fermions, describing the charge degrees of freedom, and a ground state wave function for a "squeezed Heisenberg chain", i.e. a chain where all empty sites are "squeezed out", describing the spin degrees of freedom. For a finite number of electrons, the pseudomomenta of the spinless fermions are related to the "spin rapidity" of the spin degrees freedom, but in the thermodynamic limit the distribution of the pseudomomenta becomes a constant [4], i.e., that of a free spinless fermion system. For the large U Hubbard model and for the $t-J$ model at $t=J$, the large scale behaviour of several correlation functions have been computed combining Luttinger–liquid [5] and conformal field theory [6] techniques. Related results for $t >> J$ have also been obtained in [7], by means of a more standard quantum field-theory approach, using a special mean field treatment of the non-local, or "string" field operators. Typical contributions for an euclidean two–point functions are found to be of the form

$$\frac{e^{i\frac{\pi}{2}\rho n}}{(x-iv_ct)^{\alpha_c^-}(x+iv_ct)^{\alpha_c^+}(x-iv_st)^{\alpha_s^-}(x+iv_st)^{\alpha_s^+}} \tag{1.2}$$

for suitable $n \in \mathbf{Z}, \alpha_c^\pm, \alpha_s^\pm \in \mathbf{Q}$, where ρ is the electron density, v_c and v_s are

the charge and spin velocities, respectively. Hence the correlation functions also exhibit charge–spin separation. These results show that the key features of the 1D model can be understood in terms of two low–energy excitations, the holon, charged but spinless and the spinon, neutral with spin $\frac{1}{2}$.

Partly on the basis of analogy with the 1D model, Anderson conjectured [8] that the physics of the 2D model can also be understood in terms of low energy excitations, charged but spinless (holons) and neutral with spin $\frac{1}{2}$ (spinons). Depending on the statistics of the holons (sometimes called "slave–particles") we have the slave–fermion [9], slave–boson [10] and the more exotic slave–semion approach, advocated by Laughlin [11] (generalized to a slave–anyon approach in [12]). The semions are special kind of anyons (see e.g. [13]), i.e. excitations obeying abelian braid statistics, with statistics parameter $\theta = \frac{1}{4}$, i.e. the exchange of the field operators creating such excitations produces a phase factor $\pm i$, instead of +1 or -1 characterizing bosons and fermions, respectively.

Recently, a bosonization scheme for two–dimensional fermionic systems has been proposed, based on the introduction of an abelian Chern–Simons gauge field [14]. Such a scheme has been extended to a non–abelian version in [15] and both versions have been applied to the $t-J$ model in [16] (see also [17,12]). The $U(1)$ Chern–Simons bosonization has been shown to correspond essentially to the slave–boson and slave–fermion approaches (depending on the choice of the gauge fixing [16]); while the non–abelian $U(1)\times SU(2)$ Chern–Simons bosonization corresponds to the slave semion–approach.

Although every bosonization scheme yields an exact identity between correlation functions of the original fermionic field and suitable bosonic correlation functions, the mean field approximation (MFA) gives different results in different bosonization schemes. It is then natural to ask which one of these schemes has a better chance to describe correctly in MFA the physics of the model. One expects that possible indications for the answer might be obtained from comparison with the known analytical results of the 1D $t-J$ model.

One is then naturally led to discuss a dimensional reduction of Chern–Simons bosonization to 1D systems. One can verify that this reduction corresponds, roughly speaking, to a Jordan–Wigner–like bosonization for the gauge group $U(1)$ and a suitable non–abelian generalization of it for the gauge group $U(1) \times SU(2)$.

In this paper we show that the large distance behaviour of the correlation functions of the 1D model are indeed reproduced by a mean field theory of the $U(1) \times SU(2)$ bosonization; thus, interpreting spinon and holon fields as the 1D

counterparts of semion fields, i.e. they obey abelian braid statistics with statistics parameter $\theta = \frac{1}{4}$ (in 1D only the statistics of fields, but not those of excitations are well defined, see, e.g.,[18].) It turns out that this dimensional reduction gives essentially a more systematic justification and refined structure to the approach followed in [7].

This result shows how to obtain the features of the 1D model in terms of standard quantum field theory techniques, and encourages us to pursue the study of the non–abelian Chern–Simons bosonization of the 2D $t-J$ model, suggesting some ideas for developing a possibly reasonable mean–field treatment. Furthermore, it implicitly supports the interpretation of spinons and holons of the 2D model as semions, if they are still well defined excitations.

The plan of the paper is the following:

– in sect. 2 we outline the Chern–Simons bosonization scheme and apply it to the 2D $t-J$ model;

– in sect. 3 we perform the dimensional reduction to the 1D model for the partition function;

– in sect. 4 we discuss the mean–field approximation;

– in sect. 5 we perform the dimensional reduction of correlation functions and discuss their mean–field treatment.

Some detailed computations are deferred to Appendices.

2. The Chern–Simons bosonization

We recall the main definitions and results of Chern–Simons bosonization scheme applied to spin $\frac{1}{2}$ fermion systems in 2D [15,16].

Let $\Psi_\alpha, \Psi^*_\alpha$ (resp. $\Phi_\alpha, \Phi^*_\alpha), \alpha = 1,2$ be two–component Grassmann (resp. complex) fields describing the degrees of freedom of a spin $\frac{1}{2}$ canonical non-relativistic fermion (boson) field operator $\hat{\Psi}_\alpha$ (resp. $\hat{\Phi}_\alpha$).

Consider a system of spin $\frac{1}{2}$ non–relativistic fermions interacting via an instantaneous, spin independent two–body potential, and in the presence of an external, minimally coupled abelian gauge field A. The classical euclidean action of the system is denoted by $S(\Psi, \Psi^*|A)$. Let B (resp. V) be a $U(1)$ $\Big($resp. $SU(2)\Big)$ gauge field and let

$$
\begin{aligned}
S_{c.s.}(B) &= \frac{1}{4\pi i}\int d^3x\epsilon_{\mu\nu\rho}B^\mu\partial^\nu B^\rho(x)\\
S_{c.s.}(V) &= \frac{1}{4\pi i}\int d^3xTr\epsilon_{\mu\nu\rho}\Big(V^\mu\partial^\nu V^\rho+\frac{2}{3}V^\mu V^\nu V^\rho\Big)(x)
\end{aligned}
\tag{2.1}
$$

be the corresponding euclidean Chern–Simons actions. Then the following bosonization formulas can be derived:

1) the grand–canonical partition function of the fermion system is given by:

$$
\begin{aligned}
&\int\mathcal{D}\Psi\mathcal{D}\Psi^* e^{-S(\Psi,\Psi^*|A)}\\
\text{a)} &= \frac{\int\mathcal{D}B\int\mathcal{D}\Phi\mathcal{D}\Phi^* e^{-[S(\Phi,\Phi^*|A+B)+S_{c.s.}(B)]}}{\int\mathcal{D}Be^{-S_{c.s.}(B)}},\\
\text{b)} &= \frac{\int\mathcal{D}B\mathcal{D}V\int\mathcal{D}\Phi\mathcal{D}\Phi^* e^{-[S(\Phi,\Phi^*|A+B+V)+2S_{c.s.}(B)+S_{c.s.}(V)]}}{\int\mathcal{D}B\mathcal{D}Ve^{-[2S_{c.s.}(B)+S_{c.s.}(V)]}},
\end{aligned}
\tag{2.2}
$$

where a) corresponds to the $U(1)$–bosonization and b) corresponds to the $U(1)\times SU(2)$–bosonization and gauge fixings for the respective gauge symmetries of the actions are understood.

2) Let $\gamma_x, x=(x^0,\vec{x})$ denote a string connecting x to infinity in the x^0–euclidean time plane; then one can prove an identity between the correlation functions of $\Psi_\alpha,\Psi^*_\alpha$ in the fermionic theory and the correlation functions of the non–local fields
a)

$$
\Phi_\alpha(\gamma_x|B)=e^{i\int_{\gamma_x}B}\Phi_\alpha(x),\quad \Phi^*_\alpha(\gamma_x|B)=e^{-i\int_{\gamma_x}B}\Phi^*_\alpha(x)
$$

in the $U(1)$-bosonized theory, and
b)

$$
\begin{aligned}
\Phi_\alpha(\gamma_x|B,V) &= e^{i\int_{\gamma_x}B}(Pe^{i\int_{\gamma_x}V})_{\alpha\beta}\Phi_\beta(x),\\
\Phi^*_\alpha(\gamma_x|B,V) &= \Phi^*_\beta(x)e^{-i\int_{\gamma_x}B}(Pe^{-i\int_{\gamma_x}V})_{\beta\alpha}
\end{aligned}
\tag{2.3}
$$

in the $U(1)\times SU(2)$–bosonized theory, for spin–singlet correlation functions. $P(\cdot)$ in eq. (2.3) denotes the path–ordering, which amounts to the usual time ordering $T(\cdot)$, when "time" is used to parametrize the curve along which one integrates. (For a careful discussion of boundary conditions and further details, see [15,16]).

These bosonization formulas can be derived using a Feynman–Kac representation of the partition function and the correlation functions, expressing them in terms of brownian paths in $\mathbf{R}^2$. In this representation the only difference between

fermions and bosons are minus signs related to permutation in the order of initial and final points of the paths. Using the fact that the probability for two brownian paths in $\mathbf{R}^2$ to intersect each other at a fixed time is zero, one can prove that the configurations of brownian paths appearing in the Feynman–Kac formulas are braids with probability 1, and periodic b.c. in time convert these braids into knots. Finally, the minus signs associated with permutations and converting bosons into fermions are obtained from Chern–Simons expectation values of the exponential of the gauge fields B, V arising from their minimal coupling to the matter bosonic fields, integrated over the knots formed by the brownian path configurations, following the construction of knots invariant in Chern–Simons theory (see e.g.[19]).

As this sketchy explanation suggests, one can apply the same techniques to lattice theories, using a lattice version of Feynman–Kac formula [20,16], expressing partition function and correlation functions in terms of random walks in the lattice $\mathbf{Z}^2$, retaining the Chern–Simons gauge fields in the continuum version, and provided the two–body potential contains a <u>hard–core</u> term. In fact, the probability that two random walks intersect at a fixed time in $\mathbf{Z}^2$ is not 0, so in order for the random walk configurations to be braids with probability 1, one needs a hard–core term forbidding intersections among random walks.

As a result of this brief discussion, one can understand that the bosonization formulas can be applied to the 2D $t-J$ model because:

1) the Gutzwiller projection acts as a hard–core term;
2) introducing a Hubbard–Stratonovich complex gauge field X one can rewrite the Heisenberg term as a standard kinetic term with minimal coupling to X (view in the bosonization procedure as introducing an external gauge field) plus a two–body spin–independent potential.

In fact, the grand–canonical partition function of the $t-J$ model at temperature $T = k_B/\beta$ (where k_B is the Boltzmann constant) and chemical potential μ can be rewritten [21] as:

$$\Xi_{t-J}(\beta, \mu) = \int \mathcal{D}X\mathcal{D}X^*\mathcal{D}\Psi\mathcal{D}\Psi^* e^{-S_{t-J}(\Psi,\Psi^*,X,X^*)} \tag{2.4}$$

with

$$S_{t-J}(\Psi, \Psi^*, X, X^*) = \int_0^\beta d\tau \sum_{<ij>} \frac{2}{J} X^*_{<ij>} X_{<ij>} + [(-t + X_{<ij>})\Psi^*_{i\alpha}\Psi_{j\alpha}$$

$$+h.c.] + \sum_i \Psi^*_{i\alpha}(\frac{\partial}{\partial\tau} + \mu)\Psi_{i\alpha} + \sum_{i,j} u_{i,j}\Psi^*_{i\alpha}\Psi^*_{j\beta}\Psi_{j\beta}\Psi_{i\alpha}, \tag{2.5}$$

where the two–body potential is given by

$$u_{i,j} = \begin{cases} +\infty & i = j \\ -\frac{J}{4} & i,j \text{ n.n.} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

(The euclidean–time (τ) dependence of the fields here and after is not explicitly written, and repeated spin indices are summed over.)

The bosonized action is obtained via substituting the time derivative by the covariant time derivative and the spatial lattice derivative by the covariant spatial lattice derivative: e.g. in the $U(1) \times SU(2)$ bosonization

$$\begin{aligned} \Psi^*_{j\alpha}\frac{\partial}{\partial\tau}\Psi_{j\alpha} &\longrightarrow \Phi^*_{j\alpha}\Big[\Big(\frac{\partial}{\partial\tau} + iB_0(j)\Big)\mathbb{1} + iV_0(j)\Big]_{\alpha\beta}\Phi_{j\beta}, \\ \Psi^*_{i\alpha}\Psi_{j\alpha} &\longrightarrow \Phi^*_{i\alpha}e^{i\int_{<ij>}B}(Pe^{i\int_{<ij>}V})_{\alpha\beta}\Phi_{j\beta}. \end{aligned} \tag{2.7}$$

One can then decompose

$$\Phi_{j\alpha} = \tilde{E}_j\Sigma_{j\alpha}, \tag{2.8}$$

with the constraint

$$\Sigma^*_{j\alpha}\Sigma_{j\alpha} = 1, \tag{2.9}$$

where Σ_α and $\tilde{E}$ are a 2–component and a 1–component complex lattice fields, respectively. However, such decomposition is ambiguous, since we can still perform a local $U(1)$–gauge tranformation leaving Φ invariant:

$$\begin{aligned} \tilde{E}_j &\to \tilde{E}_j e^{i\Lambda_j}, \\ \Sigma_{j\alpha} &\to \Sigma_{j\alpha}e^{-i\Lambda_j}, \quad \Lambda_j \in [0, 2\pi). \end{aligned} \tag{2.10}$$

Therefore, the theory expressed in terms of Φ is equivalent to the theory in terms of E and Σ only with a gauge–fixing term of (2.10), not breaking the $U(1) \times SU(2)$ gauge invariance of the Φ–theory. As an example one can choose a Coulomb gauge for the $U(1) \times SU(2)$–gauge invariant field "arg $(\Sigma^*_i Pe^{i\int_{<ij>}V}\Sigma_j)$" supported on links:

$$\sum_{<ij>:\ell\in<ij>} \text{arg } (\Sigma^*_i Pe^{i\int_{<ij>}V}\Sigma_j) = 0. \tag{2.11}$$

In terms of $\tilde{E}$ and Σ the $U(1)\times SU(2)$–bosonized action of the $t-J$ model is given by

$$
\begin{aligned}
S_{t-J}(\tilde{E},\tilde{E}^*,\Sigma,\Sigma^*,X,X^*,B,V) = &\int_0^\beta d\tau \sum_{<ij>} \frac{2}{J} X^*_{<ij>} X_{<ij>} \\
&+[(-t+X_{<ij>})\tilde{E}^*_j e^{i\int_{<ij>} B} \tilde{E}_i \Sigma^*_{i\alpha} (Pe^{i\int_{<ij>} V})_{\alpha\beta}\Sigma_{j\beta} + h.c.] \\
&+\sum_j \tilde{E}^*_j \Big(\frac{\partial}{\partial\tau} - iB_0(j) + \mu + \frac{J}{2}\Big)\tilde{E}_j + \tilde{E}^*_j\tilde{E}_j\Sigma^*_{j\alpha}\Big(\frac{\partial}{\partial\tau}\mathbb{1} + iV_0(j)\Big)_{\alpha\beta}\Sigma_{j\beta} \\
&+\sum_{i,j} u_{i,j}\tilde{E}^*_i\tilde{E}_i\tilde{E}^*_j\tilde{E}_j + 2S_{c.s.}(B) + S_{c.s.}(V)
\end{aligned}
\tag{2.12}
$$

with constraint (2.9) and gauge fixings understood.

One can easily convert $\tilde{E}$ into a fermion field E, by inverting the sign of $S_{c.s.}(B)$; one then omits the hard–core term in $u_{i,j}$. (In fact to perform this transformation we couple E to a new $U(1)$ gauge field B' with action $S_{c.s.}(B')$, changing variable $B \to B'' = B + B'$ and integrating out B the claimed result follows; one then rewrites B'' again as B.) Integrating out X, X^* and using the anti–commutation properties of the field E, one obtains the action

$$
\begin{aligned}
S_{t-J}(E,E^*,\Sigma,\Sigma^*,B,V) = &\int_0^\beta d\tau \sum_j E^*_j\Big(\frac{\partial}{\partial\tau} + iB_0(j) + \mu + \frac{J}{2}\Big)E_j \\
&+E^*_j E_j \Sigma^*_{j\alpha}\Big(\frac{\partial}{\partial\tau}\mathbb{1} + iV_0(j)\Big)_{\alpha\beta}\Sigma_{j\beta} \\
&+\sum_{<ij>}(-tE^*_j e^{i\int_{<ij>} B} E_i \Sigma^*_{j\alpha}(Pe^{i\int_{<ij>} V})_{\alpha\beta}\Sigma_{i\beta} + h.c.) \\
&+\frac{J}{2}E^*_j E_j E^*_i E_i \Big\{|\Sigma^*_{i\alpha}(Pe^{i\int_{<ij>} V})_{\alpha\beta}\Sigma_{j\beta}|^2 - \frac{1}{2}\Big\} - 2S_{c.s.}(B) + S_{c.s.}(V).
\end{aligned}
\tag{2.13}
$$

Notice that if instead of E we use the bose field $\tilde{E}$, the J term gets an opposite sign due to the commutation properties of $\tilde{E}$.

In terms of these variables the correlation functions of Ψ_α are given in the bosonized theory by

$$
\Phi_\alpha(\gamma_x|B,V) = e^{i\int_{\gamma_x} B} E_x (Pe^{i\int_{\gamma_x} V})_{\alpha\beta}\Sigma_{x\beta}. \tag{2.14}
$$

Therefore one can view our original electron field as a product of $U(1)\times SU(2)$–gauge invariant fields $E(\gamma_x|B) \equiv e^{i\int_{\gamma_x} B} E_x$ (holon) and $\Sigma_\alpha(\gamma_x|V) =$

$(Pe^{i\int_{\gamma_x} V})_{\alpha\beta}\Sigma_{x\beta}$ (spinon). From the coefficient of the Chern–Simons terms of B and V one can derive [15] that the corresponding non local holon and spinon field operators obey semionic statistics, i.e. their statistics parameter is $\theta = \frac{1}{4}$.

3. Reduction to 1D: the partition function

The reduction of the 2D–system to 1D is obtained by letting the electron field sit on the lattice $\mathbf{Z}$ instead of $\mathbf{Z}^2$. Denote by 1 the spatial dimension in $\mathbf{R}^2$ along the 1D lattice $\mathbf{Z}$.

Since in the partition function the only dependence of B_2 and V_2 is in the Chern–Simons action, one can integrate them out yielding the constraints

i)

$$\epsilon_{\mu\nu}\partial^\mu B^\nu = 0,$$

ii)

$$\epsilon_{\mu\nu}(\partial^\mu V^\nu + V^\mu V^\nu) = 0 \quad ; \quad \mu, \nu = 0, 1. \tag{3.1}$$

The Coulomb gauge–fixing condition for the $U(1)$ gauge invariance, $B_1 = 0$, with the boundary condition $B_\mu(x^1 = +\infty) = 0$, together with the constraint i) yields $B_\mu = 0$.

The constraint ii) is solved by

$$iV_\mu(x) = g^\dagger(x)\partial_\mu g(x), \qquad g(x) \in SU(2) \tag{3.2}$$

and in terms of g and Σ the $SU(2)$ gauge transformation reads as

$$g_j \to r_j g_j, \qquad \Sigma_j \to r_j \Sigma_j, \Sigma^*_j \to \Sigma^*_j r^\dagger_j, \tag{3.3}$$

where $r_j \in SU(2)$ and $g_j(\tau) \equiv g(\tau, j)$.

To gauge–fix (3.3) we use a ferromagnetic reference spin configuration, by setting

$$\Sigma_j = \binom{1}{0}. \tag{3.4}$$

Finally the gauge–fixing (2.11) reads, being reduced to 1D:

$$\arg\,(\Sigma^*_i g^\dagger_i g_j \Sigma_j) = \delta. \tag{3.5}$$

Choosing the constant δ in (3.5) to be 0 for later convenience, in the gauge (3.4), this implies that $(g_i^\dagger g_j)_{11}$ is real and positive, hence $g_i^\dagger g_j$ is in the linear span of $\mathbb{1}, \sigma_x, \sigma_y$, because $g_i^\dagger g_j \in SU(2)$.

Let us count the degrees of freedom (d.f.): the matter fields $E, E^*, \Sigma_\alpha, \Sigma_\alpha^*$ have $2 + (4-1)$ d.f., the field g has 3 d.f. and the gauge fixings (3.5) and (3.4) eliminate 1+3 d.f., so we are left exactly with 4 d.f. as we had for $\Psi_\alpha, \Psi_\alpha^*$; the charge degrees of freedom are carried by E, E^* , while the spin degrees of freedom are carried by g.

The partition function of the 1D $t-J$ model can be rewritten as

$$\Xi_{t-J} = \int \mathcal{D}g\mathcal{D}E\mathcal{D}E^* e^{-S_{t-J}(E,E^*,g)} \prod_{<ij>} \delta(\arg\ (g_i^\dagger g_j)_{11}) \tag{3.6}$$

with

$$\begin{aligned} S_{t-J} = &\int_0^\beta d\tau \sum_j E_j^* \Big(\frac{\partial}{\partial \tau} + \mu + \frac{J}{2} + (g_j^\dagger \partial_0 g_j)_{11} \Big) E_j \\ + &\sum_{<ij>} (-t E_j^* E_i (g_j^\dagger g_i)_{11} + h.c.) + \frac{J}{2} E_j^* E_j E_i^* E_i \Big\{ |(g_j^\dagger g_i)_{11}|^2 - \frac{1}{2} \Big\}. \end{aligned} \tag{3.7}$$

[We used $\int_{S^1 \times \mathbf{R}^2} (g^{-1} dg)^3 = 0$].

We now find a configuration, $g^m(E, E^*)$ of g minimizing the action for a fixed configuration of the holons described by E. The idea behind is that we can then treat spin fluctuations around $g^m(E, E^*)$ in some MFA. To find $g^m(E, E^*)$ we use the "Feynman–Kac" random–walk representation for the E–path integration. It can be considered as a lattice version [16, 20] of the representation of a Feynman path–integration over non–relativistic fermion fields in terms of path–integrals over trajectories of a fixed number of Fermi particles summed over all possible number of particles (see e.g.[22]).

More precisely, let Δ denote the 1D lattice laplacian defined on a scalar lattice field f by

$$(\Delta f)_i = f_{i+1} + f_{i-1} - 2 f_i,$$

let $d\mu(\omega)$ denote the measure on the random walks ω on the 1D lattice such that

$$(e^{-\beta \Delta})_{ij} = \int_{\substack{\omega(0)=i \\ \omega(\beta)=j}} d\mu(\omega), \quad \beta > 0, \tag{3.8}$$

let P_N be the group of permutations of N elements and for $\pi \in P_N$ let $\sigma(\pi)$ denote the order of the permutation. Then the partition function of the E–system in a fixed g configuration can be written as:

$$Z(g) \equiv \int \mathcal{D}E\mathcal{D}E^* e^{-S_{t-J}(E,E^*,g)} =$$

$$= \sum_{N=0}^{\infty} \frac{e^{\beta(\mu+\frac{J}{2})N}}{N!} \sum_{\pi\in P_N} (-1)^{\sigma(\pi)} \sum_{j_1,\ldots,j_N} \int_{\substack{\omega_r(0)=j_r \\ \omega_r(\beta)=j_{\pi(r)}}} \prod_{r=1}^{N} d\mu(\omega_r)\cdot$$

$$\Big[\prod_{r=1}^{N} \prod_{<ij>\in\omega_r^{\perp}} t(g_i^{\dagger}g_j)_{11} \exp\Big\{\int_{\omega_r^{\parallel}} d\tau (g^{\dagger}\partial_\tau g)_{11}\Big\}\cdot$$

$$\exp\Big\{-\sum_{r,j=1}^{N} \frac{J}{2}\int_0^{\beta} d\tau \delta_{|\omega_r(\tau)-\omega_j(\tau)|,1}[|(g^{\dagger}(\omega_{r(\tau)})g(\omega_j(\tau)))_{11}|^2 - \frac{1}{2}]\Big\}\Big], \tag{3.9}$$

where $\omega_r^{\perp}$ (resp. $\omega_r^{\parallel}$) denotes the component of ω_r perpendicular (resp. parallel) to the time axis. We first notice that, due to the Pauli principle in 1D, only the trivial permutation contributes to (3.9), since the random walks cannot intersect each other.

Consider a fixed configuration of random walks $\underline{\omega} = \{\omega_1, \ldots\omega_N\}$; using the inequality $|(g_i^{\dagger}g_j)_{11}| \leq 1$ and $Re(g^{\dagger}\partial_\tau g)_{11} = 0$, one can bound from above the absolute value of its contribution to (3.9) in square bracket by

$$\prod_{r=1}^{N} \prod_{<ij>\in\omega_r^{\perp}} t \ \exp\Big\{-\sum_{r,j=1}^{N}\{\frac{J}{2}\int_0^{\beta} d\tau \delta_{|\omega_r(\tau)-\omega_j(\tau)|,1}\cdot[-\frac{1}{2}]\}\Big\}.$$

This upper bound is exactly saturated by a configuration of g satisfying

$$(g^{\dagger}\partial_\tau g)_{11} = 0 \qquad \text{on} \quad \omega_r^{\parallel}, \tag{3.10}$$

$$(g_i^{\dagger}g_j)_{11} = e^{i\delta_{<ij>}}, \quad \delta_{<ij>} = -\delta_{<ji>}, \quad \text{if} < ij > \in \omega_r^{\perp}, \tag{3.11}$$

$$\Big(g^{\dagger}(\omega_r(\tau)g(\omega_j(\tau)\Big)_{11} = 0 \qquad \text{if} \quad |\omega_r(\tau) - \omega_j(\tau)| = 1. \tag{3.12}$$

(The arbitrariness in $\delta_{<ij>}$, independent on time, follows from

$$\sum_{<ij>\in\omega_r^{\perp}} \delta_{<ij>} = 0,$$

by periodicity in the time direction.)

If we further want to satisfy on $\omega_r^\perp$ the constraint

$$\arg\ (g_i^\dagger g_j)_{11} = 0, \tag{3.13}$$

we must set $\delta_{<ij>} = 0$.

To sum up, we obtain the following minimizing configuration $g^m \equiv g^m(\underline{\omega})$: we choose g^m constant during the period when the particle does not jump, to satisfy (3.10); while in a link $< ij >$ using the representation

$$g_j^{m\dagger} g_i^m = \cos\ \theta \mathbb{1} + i \text{sen} \theta \vec{\sigma} \cdot \vec{n}, \tag{3.14}$$

we see that (3.12) is satisfied by choosing $\theta = \frac{\pi}{2}, n_z = 0$ on links not in $\underline{\omega}^\perp$, whereas for links in $\underline{\omega}^\perp$ we choose $\theta = 0$ to satisfy (3.11). Therefore, we can represent the minimizing configuration as

$$g_j^m(\tau|\underline{\omega}) = e^{i\frac{\pi}{2}\sigma_x \sum_{\ell<j} \sum_{r=1}^N \delta_{\omega_r(\tau),\ell}}. \tag{3.15}$$

(At $\theta = \frac{\pi}{2}$, different choices of n_x, n_y, but satisfying the condition $n_x^2 + n_y^2 = 1$, give the same element of $SU(2)$ in (3.14); we choose to work with $n_x = 1, n_y = 0$.)

Reexpressed in terms of the fields E, E^* and denoted by $g^m(E, E^*)$ the minimizing configuration (3.15) becomes

$$g_j^m(\tau|E, E^*) = e^{i\frac{\pi}{2}\sigma_x \sum_{\ell<j} E_\ell^* E_\ell(\tau)}. \tag{3.16}$$

(We used here the left–continuity of the paths ω [20], so that at a jumping time $\omega_r(\tau) = \lim_{\epsilon \searrow 0} \omega_r(\tau + \epsilon)$, and the sign $\lim_{\epsilon \searrow 0}$ means $\epsilon \to 0$ from the positive side.)
It is natural to perform first the integration over g by changing variable from the $SU(2)$–valued field g to an $SU(2)$–valued field U describing fluctuations around the minimizing configuration g^m and defined by

$$g = U g^m. \tag{3.17}$$

For $j \in \mathbf{Z}$ we set

$$[j] = \begin{cases} 1 & j \text{ odd}, \\ 2 & j \text{ even}, \end{cases} \tag{3.18}$$

and

$$\tilde{j}(\tau|\underline{\omega}) = \sum_{\ell<j} \sum_{r=1}^{N} \delta_{\omega_r(\tau),\ell}. \tag{3.19}$$

To simplify the notation with the meaning being clear from the context, we set $\tilde{j}(\tau|\underline{\omega}) = \tilde{j}(\tau)$, so that, e.g., the r.h.s. of eq.(3.15) reads $e^{i\frac{\pi}{2}\sigma_x \tilde{j}(\tau)}$. Then, in terms of U and the random walks $\underline{\omega}$, the partition function is given by

$$\Xi_{t-J} = \sum_{N=0}^{\infty} \frac{e^{\beta(\mu+\frac{J}{2})N}}{N!} \sum_{j_1,\ldots,j_N} \int_{\substack{\omega_r(0)=j_r \\ \omega_r(\beta)=j_r}} \prod_{r=1}^{N} d\mu(\omega_r) \int \mathcal{D}U$$

$$\prod_{<ij>\in\omega_r^{\perp}} \delta\Big(\arg\,(U_i^{\dagger}U_j)_{[\tilde{i}][\tilde{j}]}\Big) \prod_{<ij>\in\omega_r^{\perp}} t(U_i^{\dagger}U_j)_{[\tilde{i}][\tilde{j}]} \exp \int_{\omega_r^{\parallel}} d\tau (U^{\dagger}\partial_\tau U)_{[\tilde{\omega}_r(\tau)][\tilde{\omega}_r(\tau)]} \cdot$$

$$\exp\Big\{-\sum_{j,r=1}^{N} \frac{J}{2} \int_0^{\beta} d\tau\ \delta_{|\omega_r(\tau)-\omega_j(\tau)|,1} \big[|(U^{\dagger}(\omega_r(\tau))U(\omega_j(\tau)))_{[\tilde{\omega}_r(\tau)][\tilde{\omega}_j(\tau)]}|^2 - \frac{1}{2}\big]\Big\}. \tag{3.20}$$

Analogously, defining

$$\tilde{j}(\tau|E,E^*) = \sum_{\ell<j} E_\ell^* E_\ell(\tau), \tag{3.21}$$

we have in terms of E, E^*, U:

$$\Xi_{t-J} = \int \mathcal{D}E\mathcal{D}E^* \int \mathcal{D}U e^{-S_{t-J}(U,E,E^*)} \prod_{<ij>} \delta(\arg\,(U_j^{\dagger}U_1)_{[\tilde{j}][\tilde{i}]} \tag{3.22}$$

with

$$S_{t-J}(U,E,E^*) = \int_0^{\beta} d\tau \sum_j E_j^*(\frac{\partial}{\partial\tau} + \mu + \frac{J}{2} + (U^{\dagger}\partial_\tau U)_{[\tilde{j}][\tilde{j}]})E_j +$$

$$\sum_{<ij>} \Big[(-tE_j^*E_i)(U_j^{\dagger}U_i)_{[\tilde{j}][\tilde{i}]} + h.c.\Big] + \frac{J}{2}E_j^*E_jE_i^*E_i\{|(U_j^{\dagger}U_i)_{[\tilde{j}][\tilde{i}]}|^2 - \frac{1}{2}\}. \tag{3.23}$$

Notice that up to now no approximations have been made and (3.20), (3.22–23) are exact rewritings of the partition functions of the 1D $t-J$ model.

4. Mean field approximation

We now wish to compare our result (3.20) with the Bethe–Ansatz ground state wave function of the $U \sim +\infty$ Hubbard model, essentially equivalent to the $t-J$ model at $J \sim +0$. To make the comparison we first restrict ourselves to

$T \sim 0$, finite volume V and finite number of electrons N, with V, N being large and we assume $\rho \equiv N/V = 1 - \delta, \delta << 1$ and $t >> J$.

Then we carry out the following mean–field treatment:

1) We assume that the spin fluctuations can be treated in mean field in the hopping term of the charged particles and we denote by t_R the renormalized hopping;

2) Since the motion of the charged particles is much faster than the spin motion, we replace $\omega_r(\tau)$ by its average in time, and since the paths cannot overlap $< \omega_r(\tau) >= \frac{r}{1-\delta}$. So we replace for the spin motion the original chain by a "squeezed chain" of lattice spacing $(1-\delta)^{-1}$ and accordingly we replace $\delta_{|\omega_r(\tau)-\omega_j(\tau)|,1}$ by its mean value $(1-\delta)\delta_{|r-j|,1}$ in the "squeezed chain". The corresponding renormalized spin coupling constant is denoted by J_R.

After making these approximations the canonical partition function decouples into a product of the partition function for a free charged holon system on the original lattice and the partition function for a spin $\frac{1}{2}$ quantum Heisenberg chain on the "squeezed lattice" with lattice spacing $(1-\delta)^{-1}$:

$$Z_{M.F.}(N) = \int \mathcal{D}E\mathcal{D}E^* e^{-\int d\tau \sum_i [E_i^* \partial_\tau E_i - \sum_{<ij>} t_R (E_i^* E_j + h.c.)}$$
$$\delta(\sum_i E_i^* E_i - N) \cdot \int^s \mathcal{D}U \delta(\arg(U_i^\dagger U_j)_{[i][j]})$$
$$e^{-\int d\tau \sum_j \{(U_j^\dagger \partial_\tau U_j)_{[j][j]} + \sum_{<ij>} \frac{J_R}{2} \{|(U_i^\dagger U_j)_{[i][j]}|^2 - \frac{1}{2}\}}, \tag{4.1}$$

where $\int^s$ means that the variables integrated over belong to the "squeezed lattice". (The restriction to the lattice of finite volume V is understood.) Therefore, this MFA correctly reproduces the features of the Bethe–Ansatz ground state wave function outlined in the Introduction.

After making these MFAs one can then take again the thermodynamic limit of the system with a fixed density of holes. Correlation functions of fields will be discussed in the next section in this limit.

To verify that the U–system in (4.1) is actually the quantum Heisenberg chain one can use the CP^1 representation of U:

$$U_j = \begin{pmatrix} b_{j\uparrow} & -b_{j\downarrow}^* \\ b_{j\downarrow} & b_{j\uparrow}^* \end{pmatrix} \equiv \begin{pmatrix} b_{j1} & -b_{j2}^* \\ b_{j2} & b_{j1}^* \end{pmatrix}, \tag{4.2}$$

where b_α is a complex 2–component field constrained by

$$b_{j\alpha}^* b_{j\alpha} = 1. \tag{4.3}$$

In fact, setting

$$\begin{pmatrix} \tilde{b}_{j1} \\ \tilde{b}_{j2} \end{pmatrix} = U_j e^{i\frac{\pi}{2}\sigma_x j} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.4}$$

the U–action becomes

$$S(\tilde{b}, \tilde{b}^*) = \int d\tau \sum_i \tilde{b}^*_{i\alpha} \partial_\tau \tilde{b}_{i\alpha} + \frac{J_R}{2} \sum_{<ij>} \tilde{b}^*_{i\alpha} \vec{\sigma}_{\alpha\beta} \tilde{b}_{i\beta} \cdot \tilde{b}^*_{j\gamma} \vec{\sigma}_{\gamma\delta} \tilde{b}_{j\delta}. \tag{4.5}$$

One recognizes in (4.5) the action of the quantum Heisenberg chain in the Schwinger boson representation. The gauge–fixing condition for U becomes then

$$\arg\ \tilde{b}^*_{i\alpha} \tilde{b}_{j\alpha} = 0, \tag{4.6}$$

and it can be seen as a gauge–fixing condition for the $U(1)$–gauge transformation

$$\tilde{b}_{j\alpha} \to \tilde{b}_{j\alpha} e^{i\zeta_j} \quad , \quad \tilde{b}^*_{j\alpha} \to \tilde{b}^*_{j\alpha} e^{-i\zeta_j} \quad , \quad \zeta_j \in [0, 2\pi). \tag{4.7}$$

(Notice that the invariance of the term with time derivative is guaranteed by the periodic b.c. in time and the constraint (4.6).)

One can go a step further by using the identity $\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\alpha\delta}$ and then treating the quartic $\tilde{b}$ term in the Gor'kov decoupling approximation. Assuming translational invariance and taking into account the $\tilde{b}$–gauge fixing, this yields

$$\tilde{S}_{MF}(\tilde{b}^*, \tilde{b}) = \int d\tau \sum_j \tilde{b}^*_{j\alpha} \partial_\tau \tilde{b}_{j\alpha} + \sum_{<ij>} \tilde{J}_R (\tilde{b}^*_{j\alpha} \tilde{b}_{i\alpha} + h.c.), \tag{4.8}$$

where $\tilde{J}_R =< \tilde{b}^*_{i\alpha} \tilde{b}_{j\alpha} > J_R$.

In the Appendix A we show that this system is equivalent to a system of spin $\frac{1}{2}$ free fermions, described by two component Grassmann fields f_α, f^*_α, Gutzwiller projected and the large–scale properties of the quantum Heisenberg chain are exactly reproduced by applying the standard abelian bosonization to this mean–field theory. The bosonization is performed in terms of a real scalar field ϕ_- with (euclidean) Thirring–Luttinger action

$$S(\phi_-) = \frac{1}{4\pi} \int [(\partial_0 \phi_-)^2 + v_s^2 (\partial_1 \phi_-)] \tag{4.9}$$

and with the following bosonization formulas for the left and right movers of the Gutzwiller projected fermions:

$$
\begin{aligned}
f_1^R(x) =& f_2^{*R}(x) \sim (2\pi)^{-\frac{1}{4}} D_-(x, \frac{1}{2}) : e^{-\frac{i}{2}\phi_-(x)} :, \\
f_2^R(x) =& f_1^{*R}(x) \sim (2\pi)^{-\frac{1}{4}} D_-(x, -\frac{1}{2}) : e^{\frac{i}{2}\phi_-(x)} :, \\
f_1^L(x) =& f_2^{*L}(x) \sim (2\pi)^{-\frac{1}{4}} D_-(x, \frac{1}{2}) : e^{\frac{i}{2}\phi_-(x)} :, \\
f_2^L(x) =& f_1^{*L}(x) \sim (2\pi)^{-\frac{1}{4}} D_-(x, -\frac{1}{2}) : e^{-\frac{i}{2}\phi_(x)} :,
\end{aligned}
\tag{4.10}
$$

where $v_s \sim \tilde{J}_R$ is the spin velocity and $D_-(x, \pm\frac{1}{2})$ is a disorder field, see [23] and Appendix A.

An interesting consequence of eq.(4.10) and (A.11) is that the spin $\frac{1}{2}$ Gutzwiller projected fermion operator $\hat{f}$ reconstructed from the (euclidean) field f does not obey fermionic commutation relations but rather"semionic" with statistics parameter $\theta = \frac{1}{4}$. In fact, let $x_\pm^\epsilon = (\epsilon, x^1), x^1 \gtrless 0$, let $\mathcal{F}_+(\mathcal{F}_-)$ be a polynomial of exponentials of f and disorder fields with support in $x^0 > \delta > \epsilon$ ($x^0 < -\delta < -\epsilon$, resp.), then (see Appendix A)

$$
\begin{aligned}
&\lim_{\epsilon \searrow 0} \langle \mathcal{F}_- f_1^R(0) f_1^R(x_\pm^\epsilon) \mathcal{F}_+ \rangle = \lim_{\epsilon \searrow 0} e^{-\frac{i}{4}[\arg\ (x_\pm^\epsilon) + \arg\ (-x_\pm^\epsilon)]} e^{\frac{i}{4}[\arg\ (x_\pm^{-\epsilon}) + \arg\ (x_\pm^{-\epsilon})]}. \\
&\langle \mathcal{F}_- f_1^R(0) f_1^R(x_\pm^{-\epsilon}) \mathcal{F}_+ \rangle = e^{\mp\frac{1}{4}\pi} e^{\pm\frac{i}{4}3\pi} \lim_{\epsilon \searrow 0} \langle \mathcal{F}_- f^R(0) f^R(x_\pm^{-\epsilon}) \mathcal{F}_+ \rangle
\end{aligned}
\tag{4.11}
$$

and similar results for f^L. A standard result (see e.g. [24]) of axiomatic quantum field theory then gives the equal-time commutation relations

$$
\hat{f}(x^1)\hat{f}(y^1) = e^{\pm\frac{i\pi}{2}} \hat{f}(y^1)\hat{f}(x^1), \qquad x^1 \gtrless y^1. \tag{4.12}
$$

5. Reduction to 1D: correlation functions

In this section we discuss the dimensional reduction to 1D for the correlation functions of the $U(1)\times SU(2)$ bosonized $t-J$ model. In the MFA discussed in sect. 4 one obtains exactly the large–scale behaviour of the correlation functions of the 1D $t-J$-model as derived in [5,6] by means of Luttinger liquid and conformal field-theory techniques. Furthermore, a simple interpretation of these results emerges in terms of two elementary excitations, the charged spinless holon and the spin $\frac{1}{2}$ neutral spinon: the electron field operator can be decomposed into a product of non local holon and spinon field operator, both obeying abelian braid statistics with statistics parameter $\theta = \frac{1}{4}$. Therefore, it is natural to view (althought not

compulsory due to the ambiguity of excitation statistics appearing in 1D) both holons and spinons as 1D analogues of semions, or, using a language more accepted in 1D (Zamolodchikov)–parafermions of order 4 [25].

We start by noticing that one should choose carefully the curve γ_x, needed in (2.14) to define the bosonized electron field, in order to have a good dimensional reduction of correlation functions. In fact as mentioned in sect. 2, Chern–Simons bosonization is well defined only if there are no intersections in the paths on which the Chern–Simons gauge fields are integrated over. This was automatically ensured by the Gutzwiller projection for the random walks representing the virtual worldlines of particles, appearing in the partition function, but if we choose e.g. γ_x as a straight line in the 1–direction γ_x may intersect many of these worldlines making the bosonization procedure ill–defined. This can be avoided by choosing γ_x as the path in $\mathbf{R}^3$ given by the union of the straight line going from $x = (x^0, x^1, x^2 = 0)$ to $x^\epsilon = (x^0, x^1, x^2 = \epsilon)$ and the straight line joining x^ϵ to $-\infty$ in the 1–direction. One then takes the limit $\epsilon \searrow 0$.

We discuss a general bosonization formula for fermion fields $\Psi_\alpha, \Psi^*_\alpha$, but one should keep in mind that our formulas apply only if the fermion indices appearing in the expectation values are saturated in a spin–singlet combination, since the non–abelian Chern–Simons bosonization formulas have been proved only for spin–singlet correlations[16]. Of course, we may use the global $SU(2)$ invariance of the $t-J$ model action to reduce more general correlations to linear combinations of spin–singlet correlations, e.g., $\langle \Psi^*_{x^1\mu}(x^0)\Psi_{y^1\nu}(y^0)\rangle = \frac{\delta_{\mu\nu}}{2}\langle \Psi^*_{x^1\alpha}(x^0)\Psi_{y^1\alpha}(y^0)\rangle$.

With a spin–singlet arrangement of indices understood and assuming $x^0_r < x^0_{r+1} < y^0_s < y^0_{s+1}$, $r, s = 1, ..., n-1$, the non–vanishing $2n$–point fermion correlation functions are given in terms of E, B, Σ, V, by

$$
\begin{aligned}
&\langle \prod_{r=1}^{n} \Psi^*_{x^1_r\alpha_r}(x^0_r) \prod_{s=1}^{n} \Psi_{y^1_s\alpha_s}(y^0_s)\rangle = \\
&= \langle \prod_{r=1}^{n} E^*_{x^1_r}(x^0_r) \exp\{-i\int_{\gamma_{x_r}} B\}[\Sigma^*_{x^1_r}(x^0_r)(P\exp\{-i\int_{\gamma_{x_r}} V\})]_{\alpha_r} \\
&\prod_{s=1}^{n} E_{y^1_s}(y^0_s)\exp\{i\int_{\gamma_{y_s}} B\}[(P\exp\{i\int_{\gamma_{y_s}} V\})\Sigma_{y^1_s}(y^0_s)]_{\beta_s}\rangle.
\end{aligned} \tag{5.1}
$$

The dependence of B_2 and V_2 is now both in the Chern–Simons action and in the strings $\{\gamma_{x_r}, \gamma_{y_s}\}$ attached to the fields, in the part involving the infinitesimal excursion in the 2–direction, as specified above.

Let us first discuss the B_2–integration which is simpler. It yields the constraint

$$\epsilon_{\mu\nu}\partial^{\mu}B^{\nu}(z)=\pi\chi_{[0,\epsilon]}(z^2)\{\sum_s\delta(z^1-y^1_s)\delta(z^0-y^0_s)-\sum_r\delta(z^1-x^1_r)\delta(z^0-x^0_r)\} \quad (5.2)$$

solved, with the Coulomb gauge fixing $B_1=0$ and b.c. $B_\mu(x^1=+\infty)=0$ by

$$B_0(z)=\pi\chi_{[0,\epsilon]}(z^2)\{\sum_s\delta(z^0-y^0_s)\Theta(y^1_s-z^1)-\sum_r\delta(z^0-x^0_r)\Theta(x^1_r-z^1)\}, \quad (5.3)$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$ and Θ is the Heaviside step function. Evaluated with a regularization procedure, the term containing B_0 in the action is given, in the presence of the field insertions, by

$$-\sum_r i\frac{\pi}{2}\sum_{\ell<x^1_r}E^*_\ell E_\ell(x^0_r)+\sum_s i\frac{\pi}{2}\sum_{\ell<y^1_s}E^*_\ell E_\ell(y^0_s) \quad (5.4)$$

and this is the only contribution in the correlation functions due to the field B. (The appearence of the factor $\frac{1}{2}$ w.r.t. (5.3) is due to the fact that the lattice points ℓ giving non–vanishing contributions are at the boundary of the support of B_0, hence a regularization of the δ–functions involved contribute $\frac{1}{2}$ to the integral.)

Remark

Since B has its curvature concentrated at $\underline{x}=\{x_r\}$ and $\underline{y}=\{y_s\}$, one can view the contribution (5.4) in the action as the effect of a disorder field, in the spirit of refs. [23,26,27].

Let us sketch the result of the V_2–integration, and for more details see, e.g. [28]. To every field insertion we assign a spin $\frac{1}{2}$ representation of $SU(2)$ with right action for the creation and left action for the annihilation fields, while the Lie–algebra generators acting on the p–th representation are denoted by $\sigma^p_a, a=x,y,z$. Integrating out V_2 one obtains a matrix constraint analogous to (5.2):

$$\begin{aligned}&\epsilon_{\mu\nu}(\partial^{\mu}V^{\nu}+V^{\mu}V^{\nu})_a(z)=\\&2\pi\chi_{[0,\epsilon]}(z^2)\{\sum_s\delta(y^1_s-z^1)\delta(y^0_s-z^0)\frac{\sigma^s_a}{2}\sum_r\delta(x^1_r-z^1)\delta(x^0_r-z^0)\frac{\sigma^r_a}{2}\}.\end{aligned} \quad (5.5)$$

A particular matrix valued solution of (5.5) is given by

$$(\bar{V}_0)_a(z) = 2\pi\chi_{[0,\epsilon]}(z^2)\{\sum_s \delta(y_s^0 - z^0)\Theta(y_s^1 - z^1)\frac{\sigma_a^s}{2} - \sum_r \delta(x_r^0 - z^0)\Theta(x_r^1 - z^1)\frac{\sigma_a^r}{2}\},$$
$$(\bar{V}_1)_a(z) = 0. \tag{5.6}$$

The general solution of (5.5) is obtained from $\bar{V}_\mu$ by an $SU(2)$ gauge transformation. If we perform the change of variable (3.17), it is given (at $z^2 = 0$) by:

$$\begin{aligned} V_0(z) =&\Big(\exp\{-i\frac{\pi}{2}\sigma_x \sum_{\ell<z^1} E_\ell^* E_\ell(z^0)\}U^\dagger(z)\Big)[(\bar{V}_0)_a(z)\sigma_a + \partial_0]\Big(U(z)\cdot \\ &\exp\{i\frac{\pi}{2}\sigma_x \sum_{\ell<z^1} E_\ell^* E_\ell(z^0)\}\Big), \\ V_1(z) =&\Big(\exp\{-i\frac{\pi}{2}\sigma_x \sum_{\ell<z^1} E_\ell^* E_\ell(z^0)\}U^\dagger(z)\Big)\partial_1\Big(U(z)\exp\{i\frac{\pi}{2}\sigma_x \sum_{\ell<z^1} E_\ell^* E_\ell(z^0)\}\Big). \end{aligned} \tag{5.7}$$

To understand the effect of the field (5.7), we use a random walk representation (see [16,20] for details, with an erratum in [23]) for the correlation functions (5.1) at $\beta \sim +\infty$, analogous to the one appearing in (3.20) for the partition function:

$$\begin{aligned} &\langle \prod_{r=1}^n E_{x_r^1}^*(x_r^0) \prod_{s=1}^n E_{y_s^1}(y_s^0) \prod_r e^{-i\int_{\gamma_{x_r}} B}(Pe^{-i\int_{\gamma_{x_r}} V})_{1\alpha_r} \prod_s e^{i\int_{\gamma_{y_s}} B}(Pe^{i\int_{\gamma_{y_s}} V})_{\beta_s 1}\rangle \\ &=(\Xi_{t-J})^{-1} \sum_{\pi\in P_n} \prod_{k=1}^n \sum_{\ell_k=0,1\ldots} (-1)^{\ell_k} \int_{\hat{\omega}_k(y^0_{\pi(s)}+\ell_k\beta)=y^1_{\pi(s)}}^{\hat{\omega}_k(x_r^0)=x_r^1} \prod_{k=1}^n d\mu(\hat{\omega}_k) e^{(\mu+\frac{J}{2})(y^0_{\pi(s)}+\ell_k\beta - x_r^0)}\cdot \\ &\sum_{N=0}^\infty \frac{e^{\beta(\mu+\frac{J}{2})N}}{N!} \sum_{j_1,\ldots,j_N} \int_{\substack{\omega_p(0)=j_p \\ \omega_p(\beta)=j_p}} \prod_{p=1}^N d\mu(\omega_p) \int \mathcal{D}U \prod_{<ij>\in\underline{\omega}_N^\perp} \Big\{\delta(\arg\,(U_i^\dagger U_j)_{[\tilde{i}][\tilde{j}]})\cdot \\ &t(U_i^\dagger U_j)_{[\tilde{i}][\tilde{j}]}\Big\} \prod_{\omega\in\underline{\omega}_N} \exp\Big\{\int_{\omega^\parallel} d\tau (U^\dagger(\partial_\tau + \bar{V}_0)U)_{[\tilde{\omega}(\tau)][\tilde{\omega}(\tau)]}\Big\}\cdot \\ &\exp\Big\{-\sum_{\omega,\omega'\in\underline{\omega}_N} \frac{J}{2}\int_0^\beta d\tau\delta_{|\omega(\tau)-\omega'(\tau)|,1}\Big[|(U^\dagger(\omega(\tau))U(\omega'(\tau))_{[\tilde{\omega}(\tau)][\tilde{\omega}'(\tau)]}|^2 - \frac{1}{2}\Big]\Big\}\cdot \\ &\prod_r e^{i\frac{\pi}{2}\tilde{x}_r^1}\Big(e^{i\frac{\pi}{2}\tilde{x}_r^1\sigma_x} U^\dagger_{x_r^1}(x_r^0) e^{-i\frac{\pi}{2}\sum_{\ell<x_r^1}\sum_{\omega\in\underline{\omega}_N}(U_\ell\sigma_x U_\ell^\dagger)(x_r^0)\delta_{\omega(x_r^0),\ell}}\Big)_{1\alpha_r}\cdot \\ &\prod_s e^{\frac{\pi}{2}\tilde{y}_s^1}\Big(e^{i\frac{\pi}{2}\sum_{\ell<y_s^1}\sum_{\omega\in\underline{\omega}_N}(U_\ell\sigma_x U_\ell^\dagger)(y_s^0)\delta_{\omega(y_s^0),\ell}} U_{y_s^1}(y_s^0) e^{-i\frac{\pi}{2}\tilde{y}_s^1\sigma_x}\Big)_{\beta_s 1}, \end{aligned} \tag{5.8}$$

where

$$\underline{\omega}_N = \{\hat{\omega}_1, ..., \hat{\omega}_n, \omega_1, ..., \omega_N\}$$

and for $j \in \mathbf{Z}$

$$\tilde{j}(\tau) = \sum_{\ell < j} \sum_{\omega \in \underline{\omega}_N} \delta_{\omega(\tau),\ell}.$$

[The contribution of a path–ordered exponential $Pe^{i\int_{\gamma_z} V}$ has been computed by splitting γ_z into intervals between two consecutive crossing of (the projection in the $0-1$ plane of) γ_z with $\underline{\omega}$: Let $\{z_j\}_{j=1}^{p-1}$ denote the set of the spatial coordinate of the crossing points ordered from $-\infty$ to z_1, and set $z_0 = -\infty, z_p = z$. Then in (5.8) we find:

$$Pe^{i\int_{\gamma_z} V} = \prod_{j=0}^{p-1} Pe^{i\int_{z_j}^{z_{j+1}} V} = \prod_{j=0}^{p-1} \Big(e^{-i\frac{\pi}{2} j\sigma_x} U^\dagger_{z^1_j}(z^0) U_{z^1_{j+1}}(z^0) e^{i\frac{\pi}{2}(j+1)\sigma_x} \Big) =$$

$$\prod_{j=1}^{p-1} \Big(U_{z^1_j}(z^0) e^{i\frac{\pi}{2}\sigma_x} U^\dagger_{z^1_j}(z^0) \Big) U_{z^1}(z^0) e^{i\frac{\pi}{2}\tilde{z}^1\sigma_x}. \quad]$$

Apart from the presence of $\bar{V}$ and the exponentials due to the γ-strings, the key difference in (5.8) w.r.t. (3.20) is the appearence of n new random walks $\hat{\omega}_k$, starting at times x^0_r at the points x^1_r and ending in the points $y^1_{\pi(s)}$ at time $y^0_{\pi(s)}$; after wrapping $\ell_k = 0, 1...$ times around the circle of lenght β in the time direction; these paths describe the virtual worldlines of the charged particles created and annihilated by the insertions of the E and E^* fields. Using the techniques of [19,28], one can show that in the presence of $\bar{V}$ every crossing of $\underline{\omega}^{\|}_N$ with $\gamma_{x_r}(\gamma_{y_s})$ contributes to (5.8) a factor $\overset{+}{(-)} i$.

Collecting all the factors due to the field V (5.7), we derive that

$$\overset{+}{(-)} i U_j(\tau) \sigma_x U^\dagger_j(\tau) \tag{5.9}$$

is the contribution of V due to the intersection at site j and time τ of the curves $\gamma_{x_r}(\gamma_{y_s})$ with the virtual worldlines $\underline{\omega}_N$ of the charged particles described by the field E.

Combining together (5.4), (5.9) and (3.16) we find that, for the spin singlet correlations, the fermion fields can be exactly represented in terms of U, E, E^*, as

$$\Psi^*_{x^1\alpha}(x^0) = E^*_{x^1} e^{i\frac{\pi}{2}\sum_{\ell<x^1} E^*_\ell E_\ell(x^0)} \left(U^\dagger_{x^1}(x^0) e^{-i\frac{\pi}{2}\sum_{\ell<x^1} U_\ell \sigma_x U^\dagger_\ell(x^0) E^*_\ell E_\ell(x^0)}\right)_{[\tilde{x}_1]\alpha},$$

$$\Psi_{y^1\beta}(y^0) = E_{y^1}(y^0) e^{-i\frac{\pi}{2}\sum_{\ell<y^1} E^*_\ell E_\ell(y^0)} \left(e^{i\frac{\pi}{2}\sum_{\ell<y^1} U_\ell \sigma_x U^\dagger_\ell(y^0) E^*_\ell E_\ell(y^0)} U_{y^1}(y^0)\right)_{\beta[\tilde{y}^1]}. \tag{5.10}$$

To obtain a more tractable expression to compare with the results obtained by Bethe–Ansatz and Luttinger liquid techniques, one applies the MFA discussed in section 4.

An important result of the MFA is that the spin degrees of freedom appear in the "squeezed Heisenberg chain", where the Gutzwiller projection can be implemented exactly as a single occupancy constraint. The spin fluctuations $\tilde{b}_\alpha$ in the squeezed chain can be then converted into fermion fields f_α by

$$\tilde{b}_{j\alpha}(\tau) = e^{i\pi\sum_{\ell<j} f^*_{\ell\beta} f_{\ell\beta}(\tau)} f_{j\alpha}(\tau). \tag{5.11}$$

From the Gutzwiller projection we also derive, see (4.14), (4.25):

$$f^*_{j1} = e^{i\pi j} f_{j2}, \quad f^*_{j2} = e^{i\pi j} f_{j1}. \tag{5.12}$$

Denoting quantities in the squeezed chain by $[\ \cdot\ ]^s$, we obtain in MFA

$$\begin{aligned}
&\left(e^{i\frac{\pi}{2}\sum_{\ell<y^1} U_\ell \sigma_x U^\dagger_\ell(y^0) E^*_\ell E_\ell(y^0)} U_{y^1}(y^0)\right)_{\beta[\tilde{y}^1]} \underset{MFA}{\sim} \\
&\left[e^{(-)^\beta i\frac{\pi}{2}\sigma_z \sum_{\ell<y^1} b^*_{\ell1} b^*_{\ell2} + b_{\ell1} b_{\ell2}} \tilde{b}_{y^1\beta}(y^0)\right]^s \\
&= \left[e^{(-)^\beta i\frac{\pi}{2}\sigma_z \sum_{\ell<y^1} (-1)^\ell (f^*_{\ell1} f^*_{\ell2} + f_{\ell1} f_{\ell2})} e^{i\pi\sum_{\ell<y^1} f^*_{\ell\alpha} f_{\ell\alpha}(y^0)} f_{y^1\beta}(y^0)\right]^s \\
&= \left[e^{-(-)^\beta i\frac{\pi}{2}\sum_{\ell<y^1} f^*_{\ell\alpha} f_{\ell\alpha}(y^0)} f_{y^1\beta}(y^0)\right]^s.
\end{aligned} \tag{5.13}$$

To derive the large–scale properties we apply to (5.13) the abelian bosonization, obtaining at long distance, apart from an overall ultraviolet renormalization (see Appendix A):

$$\left(e^{i\frac{\pi}{2}\sum_{\ell<y^1} U_\ell \sigma_x U^\dagger_\ell(y^0) E^*_\ell E_\ell(y^0)} U_{y^1}(y^0)\right)_{1\{2\}\tilde{y}^1} \underset{MFA}{\sim} f_1^{R\{*L\}}(y). \tag{5.14}$$

(In (5.14) for brevity we introduced the following notation: if strings of the form $a\{b\}$ with mathematical symbols a, b appearing on both sides of an equation, the meaning is that the equation is valid either if we use everywhere the symbols before $\{\cdot\}$, or if we use everywhere the symbols inside $\{\cdot\}$.)

To extract the large scale properties of the charge degrees of freedom we apply the abelian bosonization also to the field E: first we rewrite it (and its conjugate) in terms of left and right movers:

$$E_j = e^{i\pi(1-\delta)j} E_j^L + e^{-i\pi(1-\delta)j} E_j^R, \tag{5.15}$$

then to the corresponding continuum fields $E^L(x), E^R(x)$ we apply the bosonization scheme rewriting their correlation functions in terms of a real scalar field, ϕ_c, with (euclidean) Thirring–Luttinger action

$$S(\phi_c) = \frac{1}{8\pi} \int d^2x [(\partial_0 \phi_c)^2 + v_c^2 (\partial_1 \phi_c)^2] \tag{5.16}$$

and with the identifications

$$E^R = D_c(x,1) : e^{\frac{i}{2}\phi_c(x)} : \text{etc.},$$

where $v_c \sim t_R$ denotes the charge velocity.

Evaluation of the $E-$string in the scaling limit gives

$$e^{\pm i \frac{\pi}{2} \sum_{\ell < x^1} E_\ell^* E_\ell(x^0)} \sim e^{\pm i \frac{\pi}{2} \int_{-\infty}^{x^1} :E^{*R} E^R + E^{*L} E^L:(y) dy} e^{\pm i \frac{\pi}{2}(1-\delta)x^1}$$
$$\sim e^{\pm \frac{i}{4}\phi_c(x) \pm i \frac{\pi}{2}(1-\delta)x^1}. \tag{5.17}$$

By the same arguments as those used in (4.11–12), one can show that the non–local field operator reconstructed from $E(x) e^{\pm i \int_{-\infty}^{x^1} :E^{*R} E^R + E^{*L} E^L:(y) dy}$ obeys an abelian braid statistics with statistics parameter $\theta = \frac{1}{4}$.

Combining together (5.12–17) we have shown that in MFA the large scale behaviour of the spin–singlet correlation functions can be derived using the identities:

$$\Psi^*_{x^1 1\{2\}}(x^0) \underset{MFA}{\sim} \Big(e^{-1\{-3\} i \frac{\pi}{2}(1-\delta)x^1} E^{*L}(x) + e^{3\{1\} i \frac{\pi}{2}(1-\delta)x^1} E^{*R}(x) \Big) \cdot$$
$$e^{+\{-\} i \frac{\pi}{2} \int_{-\infty}^{x^1} :E^{*R} E^R + E^{*L} E^L:(z) dz} f_1^{*R\{L\}}(x) \sim$$
$$D_c(x,-1)(e^{1\{3\} i \frac{\pi}{2}(1-\delta)x^1} : e^{3\{1\} \frac{i}{4}\phi_c(x)} : + e^{3\{1\} i \frac{\pi}{2}(1-\delta)x^1} : e^{-3\{-1\} \frac{i}{4}\phi_c(x)} :) \cdot$$
$$D_-(x, -\{+\}\frac{1}{2}) : e^{\frac{i}{2}\phi_-(x)} :$$

$$\Psi_{y^1 1\{2\}}(y^0) \underset{MFA}{\sim} \Big(e^{1\{3\} i \frac{\pi}{2}(1-\delta)y^1} E^L(y) + e^{-3\{-1\} i \frac{\pi}{2}(1-\delta)y^1} E^R(y) \Big) \cdot$$

$$e^{-\{+\}i\frac{\pi}{2}\int_{-\infty}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^{R\{*L\}}(y)\sim$$

$$D_c(y,+1)(e^{1\{3\}i\frac{\pi}{2}(1-\delta)y^1}:e^{-1\{-3\}\frac{i}{4}\phi_c(y)}:+e^{-3\{-1\}i\frac{\pi}{2}(1-\delta)y^1}:e^{3\{1\}\frac{i}{4}\phi_c(y)}:)\cdot$$

$$D_-(y,+\{-\}\frac{1}{2}):e^{-\frac{i}{2}\phi_-(x)}: \tag{5.18}$$

According to the general results of $U(1)\times SU(2)$ Chern–Simons bosonization, (see (3.13)), the original fermion field can be decomposed into two non–local semion fields, $U(1)\times SU(2)$–gauge invariant: $E(\gamma_x|B),\Sigma_\alpha(\gamma_x|V)$ which one may call holon and spinon field, respectively. Equation (5.18) proves that for large scales, in the MFA one can identify $E(\gamma_x|B)$ as the 1D–semion field $E(x)e^{i\int_{-\infty}^{x1}:E^{*R}E^R+E^{*L}E^L:}$ and $\Sigma_{1\{2\}}(\gamma_x|V)$ as the 1D chiral semion field $f_1^{R\{*L\}}$, a Gutzwiller projected chiral fermion field of the squeezed Heisenberg chain.

In Appendix B we show that the formulas (5.18) when applied to the correlation functions of the 1D $t-J$ model in the regime $t>>T$, indeed reproduce correctly large scale behaviours identical to those obtained with Bethe–Ansatz and Luttinger liquid techniques extrapolated from the large U Hubbard model [5] and the $t=J$, $t-J$ model [6].

To conclude, we have shown that one can obtain the correct large scale behavior of the 1D $t-J$ model by simply using a mean field theory treatment of the dimensional reduction of the $U(1)\times SU(2)$ Chern–Simons bosonization. Moreover, we have shown that the $U(1)\times SU(2)$ Chern–Simons bosonization is the most natural mathematical framework for describing the spin–charge decomposition of the electron field in terms of semionic fields. This shows that the key physical properties of the 1D $t-J$ model are captured not by an arbitrary spin–charge separation scheme (in fact slave fermion and slave boson approaches failed to reproduce the correct large scale behaviour of the correlation functions), but rather by a specific semionic form of the spin–charge separation. This gives rise to our hope that the power of this formalism and the underlying physical intuition will survive in 2D.

Appendix A

We first fermionize the system (4.8). Since the spin components of $\tilde{b}_\alpha$ are coupled via the Gutzwiller constraint, one cannot apply to them independent Jordan–Wigner transformations. To derive the correct transformation we use once more the reduction from a 2D system. Coupling the $\tilde{b}$ field to a $U(1)$ gauge field B with action $S_{c.s.}(B)$ in 2D is known to convert $\tilde{b}$ to a fermion field f. Integrating out B_0 we get the constraint

$$\epsilon_{\mu\nu}\partial^\mu B^\nu(x) = 2\pi \sum_j \tilde{b}^*_{j\alpha}\tilde{b}_{j\alpha}(x^0)\delta(x^1 - j), \quad \mu,\nu = 1,2.$$

In the Landau gauge, $\partial^\mu B_\mu = 0$, it can be solved by

$$B_\mu(x) = \sum_j \tilde{b}^*_{j\alpha}\tilde{b}_{j\alpha}(x^0)\partial_\mu \arg\,(x^1 - j).$$

Reducing this formula to 1D we find

$$B_0(x) = 0, \qquad\qquad B_1(x) = \pi \sum_j \tilde{b}^*_{j\alpha}\tilde{b}_{j\alpha}(x^0)\partial_1\Theta(x^1 - j).$$

Applying the Gutzwiller constraint we obtain

$$B_1(x) = \pi \sum_j \delta(x^1 - j),$$

so that $e^{i\int_{<ij>} B} = e^{i\pi}$ and the action for fermions reads as

$$S(f, f^*) = \int d\tau \sum_i f^*_i \partial_\tau f_i - \sum_{<ij>} \tilde{J}_R(f^*_i f_j + h.c.) \qquad (A.1)$$

with the constraint

$$f^*_{j\alpha} f_{j\alpha} = 1, \qquad (A.2)$$

i.e. it describes exactly a system of spin $\frac{1}{2}$ non-relativistic free fermions, Gutzwiller projected.

It is well known [29] that the large–scale properties of the spin $\frac{1}{2}$ quantum Heisenberg chain can be described in terms of a real scalar field φ with Luttinger–Thirring action

$$S_\xi(\varphi) = \frac{\xi}{8\pi}\int[(\partial_0\varphi)^2 + v_s^2(\partial_1\varphi)^2], \qquad (A.3)$$

where $\xi = 2$ and v_s denotes the spin velocity. (Here we use the convention in which a mass term would be described by $:\cos\ \varphi:$.)

Let us now show that we indeed recover this result starting from the mean field model (A.1). To analyze the large scale properties, following [30], we first introduce the decomposition of f in right and left movers in a lattice labelled by sites in $(2\mathbf{Z} + \frac{1}{2})$:

$$
\begin{aligned}
f_{2n\alpha} =& i^{2n} f^L_{(2n+\frac{1}{2})\alpha} + (-i)^{2n} f^R_{(2n+\frac{1}{2})\alpha}, \\
f_{2n+1\alpha} =& i^{2n+1} f^L_{(2n+\frac{1}{2})\alpha} + (-i)^{2n+1} f^R_{(2n+\frac{1}{2})\alpha}.
\end{aligned}
\tag{A.4}
$$

The Gutzwiller constraints are given by:

$$
\begin{aligned}
f^{*L}_{2n+\frac{1}{2}\alpha} f^L_{2n+\frac{1}{2}\alpha} + f^{*R}_{2n+\frac{1}{2}\alpha} f^R_{2n+\frac{1}{2}\alpha} &= 1, \\
f^{*R}_{2n+\frac{1}{2}\alpha} f^L_{2n+\frac{1}{2}\alpha} + f^{*L}_{2n+\frac{1}{2}\alpha} f^R_{2n+\frac{1}{2}\alpha} &= 0.
\end{aligned}
\tag{A.5}
$$

The fields f^L, f^R are then assumed to have a good continuum limit with linearized dispersion relations, resulting in a large–scale continuum action given, before the constraint is implemented, by

$$
S_{MF}(f, f^*) = \int d^2x \Big[f^{*R}_\alpha (\partial_0 + i v_s \partial_1) f^R_\alpha + f^{*L}_\alpha (\partial_0 - i v_s \partial_1) f^L_\alpha \Big]. \tag{A.6}
$$

Introducing two real scalar field $\phi_\alpha, \alpha = 1, 2$ and applying the standard abelian 1D bosonization (proved in [23,31] to be a special version of the duality transformation), we obtain a Luttinger–Thirring action

$$
S_{MF}(\phi_\alpha) = \sum_{\alpha=1}^{2} S_1(\phi_\alpha), \tag{A.7}
$$

and the bosonization formulas for fields [23,27]:

$$
\begin{aligned}
f^R_\alpha(x) \sim& (2\pi)^{-\frac{1}{4}} D_\alpha(x, 1) : e^{-\frac{i}{2}\phi_\alpha(x)} :, \\
f^{*R}_\alpha(x) \sim& (2\pi)^{-\frac{1}{4}} D_\alpha(x, -1) : e^{\frac{i}{2}\phi_\alpha(x)} :, \\
f^L_\alpha(x) \sim& (2\pi)^{-\frac{1}{4}} D_\alpha(x, 1) : e^{\frac{i}{2}\phi_\alpha(x)} :, \\
f^{*L}_\alpha(x) \sim& (2\pi)^{-\frac{1}{4}} D_\alpha(x, -1) : e^{-\frac{i}{2}\phi_\alpha(x)} :,
\end{aligned}
\tag{A.8}
$$

where $D_\alpha(x, \pm 1)$ is a disorder field and we adopted the notations of reference [23], to which we refer for more details. [In (A.8) $: e^{i\zeta\phi(x)} :, \zeta \in \mathbf{R}$ denotes the normal ordered exponential, defined as follows: let δ^κ_x be a regularization of the Dirac δ–function δ_x, with u.v. regulator κ, i.e. $\delta^\kappa_x \underset{\kappa\uparrow\infty}{\rightarrow} \delta_x$; then

$$
: e^{i\zeta\phi(x)} :\equiv \lim_{\kappa\uparrow\infty} e^{i\zeta\phi(\delta^\kappa_x)} (2\pi)^{\zeta^2} e^{-\frac{2\pi\zeta^2}{\xi}(\delta^\kappa_x, \Delta^{-1}\delta^\kappa_x)} \tag{A.9}
$$

with

$$
\Delta^{-1}(x, y) \equiv \frac{1}{2\pi} \ln \sqrt{(x^0 - y^0)^2 + v_s^2 (x^1 - y^1)^2}.
$$

Formally, we rewrite (A.9) as

$$: e^{i\zeta\phi(x)} := e^{i\zeta\phi(x)}(2\pi)^{\zeta^2} e^{-\frac{2\pi\zeta^2}{\xi}\Delta^{-1}(x,x)}. \quad] \qquad (A.10)$$

In the model with action (A.3) the expectation values of products of disorder fields and exponentials are given by

$$< \prod_j D(x_j, \zeta_j) e^{i\int \phi(x) f(y) dy} >$$

$$= \begin{cases} 0, \text{ if } \sum_j \zeta_j \neq 0 \text{ or } \hat{f}(0) \neq 0 \quad \text{for } f \text{ real}, \\ e^{\frac{\xi}{2}\sum_{i<j}\zeta_i\zeta_j \ln|x_i - x_j|} e^{\int d^2x d^2y f(x) \ln|x-y| f(y)} e^{i\sum_j \int d^2x f(x) \arg\ [(x^0 - x_j^0) + iv_s(x^1 - x_j^1)]} \end{cases}, \qquad (A.11)$$

where $|z - w| \equiv \sqrt{(z^0 - w^0)^2 + v_s^2(z^1 - w^1)^2}$.
One can implement the Gutzwiller constraint exactly, defining

$$\phi_\pm = \frac{\phi_1 \pm \phi_2}{2}.$$

Then, with the notations of (A.3), we have

$$S_{MF}(\phi_+, \phi_-) = S_2(\phi_+) + S_2(\phi_-) \qquad (A.12)$$

and, e.g.,

$$f_1^R(x) = D_+(x, \frac{1}{2}) D_-(x, \frac{1}{2}) : e^{-\frac{i}{2}\phi_+(x)} : e^{-\frac{i}{2}\phi_-(x)} :$$

etc, so that (taking care of the normal ordering, see [27]) the constraints (A.5) become

$$\begin{aligned} &\frac{1}{2\pi} :\partial_1\phi_+ := 0, \\ &: \cos\ (\phi_+ + \frac{\pi}{2}) :: \cos\ \phi_- := 0, \end{aligned} \qquad (A.13)$$

solved by

$$\phi_+ = \text{const}, \quad : e^{i\phi_+} := 1. \qquad (A.14)$$

As a conseguence $S_{MF}(\phi) = S_2(\phi_-)$, i.e. we recover eq. (4.9) and we obtain the bosonization formulas (4.10).

From equations (4.10) and (A.11) we derive, e.g.,

$$\langle f^*_\uparrow(x) f_\uparrow(y)\rangle = e^{i\frac{\pi}{2}(x^1-y^1)}\langle f_1^{*R}(x) f_1^R(y)\rangle + e^{-i\frac{\pi}{2}(x^1-y^1)}\langle f_1^{*L}(x) f_1^L(y)\rangle =$$
$$\frac{e^{i\frac{\pi}{2}(x^1-y^1)}}{\sqrt{(x^1-y^1)-iv_s(x^0-y^0)}} + \frac{e^{-i\frac{\pi}{2}(x^1-y^1)}}{\sqrt{(x^1-y^1)+iv_s(x^0-y^0)}}.$$

Therefore the mean field treatment of Gutzwiller projected free electrons reproduces exactly the large scale properties of the Heisenberg chain.

Remark: Suppose we keep the hard–core constraint for the individual fields $\tilde{b}_\alpha$, but perform a mean field treatment of the remaining Gutzwiller constraint, then, since the two components of $\tilde{b}_\alpha$ are not any more coupled we can fermionize them by separate Jordan–Wigner transformations. To the corresponding fermionic field f_α one can apply the same treatment as before, but since the Gutzwiller constraint disappeared, our bosonization formulas are simply eqs.(4.18). This procedure reproduces the approximate results of [7]. Therefore, the treatment of the constraint should be exact, and only afterwards one can use MFA.

Finally, let us apply equations (4.10) and (A.9) to prove (5.14): we rewrite (5.13) as

$$e^{-(-)^\beta i\frac{\pi}{2}y(1-\delta)} e^{\frac{i}{2}\phi_+(y)} D_+(y,\frac{1}{2}) D_-(y,\frac{1}{2})\cdot$$
$$\Big(e^{-i\frac{\pi}{2}(1-\delta)y^1} : e^{-\frac{i}{2}\phi_+(y)} :: e^{(-)^\beta\frac{i}{2}\phi_-(y)} : + e^{+i\frac{\pi}{2}(1-\delta)y^1} e^{\frac{i}{2}\phi_+(y)} :: e^{-(-)^\beta\frac{i}{2}\phi_-(y)} :\Big) \tag{A.15}$$

with the Gutzwiller projection implemented imposing equation (A.14). Formally, see (A.9–A.10), we have

$$e^{\frac{i}{2}\phi_+(y)} : e^{-\frac{i}{2}\phi_+(y)} :\sim e^{-\frac{1}{8}\Delta^{-1}(y,y)},$$
$$e^{\frac{i}{2}\phi_+(y)} : e^{\frac{i}{2}\phi_+(y)} :\sim e^{\frac{3}{8}\Delta^{-1}(y,y)} : e^{i\phi_+(y)} := e^{\frac{3}{8}\Delta^{-1}(y,y)}.$$

Therefore, the ratio of the coefficients of the second to the first term in (A.15) tends to 0 as the U.V. cutoff (assumed to be the inverse of the scale parameter) is removed. Recalling eq.(4.10), at large scale we recover (5.14).

Appendix B

Using (5.18) and Appendix A we compute the large scale behaviour of

1) density–density correlation function:

$$\langle : \Psi^*_\alpha \Psi_\alpha : (x) : \Psi^*_\beta \Psi_\beta : (y)\rangle \underset{MFA}{\sim} \langle : E^* E : (x) : E^* E : (y)\rangle$$

$$\sim \langle \frac{1}{2\pi} : \partial_1\phi_c : (x) \frac{1}{2\pi} : \partial_1\phi_c : (y) \rangle$$

$$+\langle : e^{i\phi_c(x)} :: e^{-i\phi_c(y)} \rangle e^{-i2\pi(1-\delta)(x^1-y^1)} + \langle : e^{-i\phi_c(x)} :: e^{i\phi_c(y)} : \rangle e^{i2\pi(1-\delta)(x^1-y^1)}$$

$$= -\frac{1}{4\pi^2}\Big[\frac{1}{((x^1-y^1)-iv_c(x^0-y^0))^2} + \frac{1}{((x^1-y^1)+iv_c(x^0-y^0))^2}\Big]$$

$$+\frac{1}{2\pi^2}\frac{\cos[2\pi(1-\delta)(x^1-y^1)]}{(x^1-y^1)^2+v_c^2(x^0-y^0)^2}; \qquad (B.1)$$

2) spin–spin correlation function

$$\langle \Psi^*_\alpha \frac{\vec{\sigma}_{\alpha\beta}}{2}\Psi_\beta(x)\Psi^*_\gamma\frac{\vec{\sigma}_{\gamma\delta}}{2}\Psi_\delta(y)\rangle =$$

$$= \frac{1}{2}\langle \Psi^*_\alpha(x)\Psi_\beta(x)\Psi^0_\beta(y)\Psi_\alpha(x)\rangle - \frac{1}{4}\langle \Psi^*_\alpha(x)\Psi_\alpha(x)\Psi^0_\beta(y)\Psi_\beta(y)\rangle \underset{MFA}{\sim}$$

$$\frac{1}{2}\langle E^*E(x)f_1^{*R}f_1^R(x)E^*E(y)f_1^*f_1^R(y)\rangle + \frac{1}{2}\langle E^*E(x)f_1^{*L}f_1^L(x)E^*E(y)f_1^{*L}f_1^L(y)\rangle$$

$$+\frac{1}{2}\langle E^*E(x)e^{-i\pi\int_{-\infty}^{x^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^Rf_1^L(x)E^*E(y)e^{i\pi\int_{-\infty}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^{*R}f_1^{*R}(y)\rangle$$

$$+\frac{1}{2}\langle E^*E(x)e^{i\pi\int_{-\infty}^{x^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^{*R}f_1^{*L}(x)E^*E(y)e^{-i\pi\int_{-\infty}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^Rf_1^L(y)\rangle$$

$$-\frac{1}{4}\langle E^*E(x)E^*E(y)\rangle \sim \frac{\delta^2}{2}\langle e^{i\pi\int_{x^1}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}\rangle e^{i\pi(1-\delta)(y^1-x^1)}\cdot$$

$$\langle f_1^{*R}(x)f_1^R(y)\rangle\langle f_1^{*L}(x)f_1^L(y)\rangle + \langle e^{-i\pi\int_{x^1}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}\rangle\cdot$$

$$e^{-i\pi(1-\delta)(y^1-x^1)}\langle f^R(x)f^{*R}(y)\rangle\langle f^L(x)f^{xL}(y)\rangle$$

$$= \delta^2\frac{\cos\,[(\pi(1-\delta)(y^1-x^1))]}{((x^1-y^1)^2+v_s^2(x^0-y^0)^2)^{\frac{1}{2}}((x^1-y^1)^2+v_c^2(x^0-y^0)^2)^{\frac{1}{4}}}, \qquad (B.2)$$

where in the third equality we use $E^*E(x) =: E^*E : (x) + \delta$ and we neglect the first term as subleading;

3) electron–electron correlation function

$$\langle \Psi^*_\mu(x)\Psi_\nu(y)\rangle = \frac{\delta_{\mu\nu}}{2}\langle \Psi^*_\alpha(x)\Psi_\alpha(y)\rangle;$$

$$\langle \Psi^*_\alpha(x)\Psi_\alpha(y)\rangle \underset{MFA}{\sim} \langle (E^{*L}(x)E^L(y)e^{-i\frac{\pi}{2}(1-\delta)(x^1-y^1)} + E^{*R}(x)E^R(y)e^{3i\frac{\pi}{2}(1-\delta)(x^1-y^1)})\cdot$$

$$e^{-i\frac{\pi}{2}\int_{x^1}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz}f_1^{*R}(x)f_1^R(y)\rangle$$

$$+\langle (E^{*L}(x)E^L(y)e^{-\frac{3}{2}i\pi(1-\delta)(x^1-y^1)} + E^{*R}(x)E^R(y)e^{i\frac{\pi}{2}(1-\delta)(x^1-y^1)})\cdot$$

$$e^{i\frac{\pi}{2}\int_{x^1}^{y^1}:E^{*R}E^R+E^{*L}E^L:(z)dz} f_1^L(x) f_1^{*L}(y)\rangle = \langle D_c(x,-1)D_c(y,1)D_-(x,-\frac{1}{2})D_-(y,\frac{1}{2})$$

$$\Big[e^{-i\frac{\pi}{2}(1-\delta)(x^1-y^1)} : e^{\frac{i}{4}\phi_c(x)} :: e^{-\frac{i}{4}\phi_c(y)} : + e^{i3\frac{\pi}{2}(1-\delta)(x^1-y^1)} : e^{-i\frac{3}{4}\phi_c(x)} :: e^{i\frac{3}{4}\phi_c(y)} : \Big]\rangle + h.c.$$

$$= \frac{1}{[(x^1-y^1)+iv_s(x^0-y^0)]^{\frac{1}{2}}} \frac{1}{[(x^1-y^1)^2+v_c^2(x^0-y^0)^2]^{\frac{1}{16}}} \cdot$$

$$\Big[\frac{e^{i\frac{\pi}{2}(1-\delta)(x^1-y^1)}}{[(x^1-y^1)+iv_c(x^0-y^0)]^{\frac{1}{2}}} + \frac{e^{i3\frac{\pi}{2}(1-\delta)(x^1-y^1)}}{[(x^1-y^1)+iv_c(x^0-y^0)]^{\frac{3}{2}}}\Big] + h.c. \qquad (B.3)$$

We see that the large scale behaviour of the correlation functions is of the form (1.2) and a comparison of the values of $n, \alpha_c^{\pm}$, $\alpha_s^{\pm}$ with those found in [5] and [6] shows that they agree exactly with the result obtained for the large U Hubbard model and the $t-J$ model at $t=J$, extrapolated to the region $t >> J$.

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